

ZARISKI STRUCTURES AND ALGEBRAIC GEOMETRY

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ABSTRACT. The purpose of this paper is to provide a new account of multiplicity for finite morphisms between smooth projective varieties. Traditionally, this has been defined using commutative algebra in terms of the length of integral ring extensions. In model theory, a different approach to multiplicity was developed by Zilber using the techniques of non-standard analysis. Here, we first reformulate Zilber's method in the language of algebraic geometry using specialisations and secondly show that, in classical projective situations, the two notions essentially coincide. As a consequence, we can recover intersection theory in all characteristics from the non-standard method and sketch the further development of the theory in connection with étale cohomology and deformation theory. The usefulness of this approach can be seen from the increasing interplay between Zariski structures and objects of non-commutative geometry, see [15].

We will work mainly in the language of Weil's Foundations, namely using varieties instead of schemes. K will denote a big algebraically closed field. $L \subset K$ will denote a small algebraically closed field. By an affine variety V , we mean a closed subset of K^n in the Zariski topology. If V is irreducible, we denote the ring of regular functions on V by $K[V]$ and the function field by $K(V)$. If $k \subset K$ is perfect, we say that V is defined over k if $I(V)$, the radical ideal of functions vanishing on V is generated by polynomials with coefficients in k . Any irreducible affine variety V has a minimal field of definition k_V with the property that any automorphism fixes V setwise iff it fixes k_V pointwise. This is a classical result due to Weil, but is in fact a special case of a more general construction due to model theorists of canonical bases, see [6]. By a variety, we will mean a set V , a covering of subsets V_1, \dots, V_m and for each i a bijection $f_i : V_i \rightarrow U_i$ with U_i an affine variety and such that for each $1 \leq i, j \leq m$, $U_{ij} = f_i(V_i \cap V_j)$ is an open subset of U_i and $f_{ij} = f_j f_i^{-1}$ is an isomorphism between the affine varieties U_{ij} and U_{ji} . A variety V then inherits a natural Zariski topology by declaring $U \subset V$ open if for each i , $f_i(U \cap V_i)$ is open in U_i . For

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$k \subset K$, we will say that V is defined over k , if the data (U_i, U_{ij}, f_{ij}) is defined over k in the sense of affine varieties. We let $P^n(K)$ denote n -dimensional projective space over K , that is K^{n+1}/\sim , where \sim is the equivalence relation on $K^{n+1} \setminus \{\bar{0}\}$ given by $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ iff $\lambda(x_0, \dots, x_n) = (y_0, \dots, y_n)$ for some $\lambda \in K$. Writing elements of $P^n(K)$ in homogenous coordinates, $(x_0 : x_1 : \dots : x_n)$, we have natural bijections f_i between K^n and $P^n(K)_i = \{\bar{x} : x_i \neq 0\}$. This gives $P^n(K)$ the structure of a variety defined over the prime subfield and an induced Zariski topology. By a projective variety V , we mean a closed subset of $P^n(K)$, using the coordinate charts f_i , V automatically is a variety in the sense defined above. Equivalently, a projective variety V is defined by a set of homogenous polynomials in $K[x_0, \dots, x_n]$ and is defined over k if the ideal $I(V)$ is generated by homogenous polynomials with coefficients in k . If a variety V is defined over k and $k \subset L \subset K$ with L algebraically closed then we will use the notation $V(L)$ to denote V considered as a variety over L . In this case, we will require that a subvariety of $V(L)$ is defined over L .

We will use the notation $X \times_Y Z$ to denote the fibre product of two varieties X and Y over Z . Given a variety V defined over k and a tuple of elements $\bar{a} \in V^n$, we will use $k(\bar{a})$ to denote the field of definition of \bar{a} . In the case when $X = \text{Spec}(L)$, corresponding to an L rational point $j : \text{Spec}(L) \rightarrow Z$;

$$\begin{array}{ccc} L \times_Z Y & \xrightarrow{i} & X \\ \downarrow pr & & \downarrow j \\ Y & \xrightarrow{f} & Z \end{array}$$

we will often use the notation $L \times_Z Y$ to denote the geometric fibre $f^{-1}(y)$ of a point $y \in Z$, considered as a variety over L . Similar notation will be used in the case of sheaves. Given varieties Y, Z and a morphism $g : Y \rightarrow Z$, we define the pullback of a coherent sheaf F on Z to be the sheafification of

$$g^*F = \mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}F$$

where $g^{-1}F(U) = \lim_{\rightarrow, g(U) \subset V} F(V)$. Again, in the case when $j : \text{Spec}(L) \rightarrow Z$ is an L rational point and F is a coherent sheaf on Z , $j^{-1}F = F_z$, the localised sheaf at z , and $L \otimes_{\mathcal{O}_{z,Z}} F_z$ is a vector space

over L which, by slight abuse of notation, corresponds to the fibre of the sheaf F at z . Given a morphism $f : X \rightarrow Y$, we let $\Omega_{X/Y}$ denote the sheaf of relative differentials on X . We will use the geometric construction of $\Omega_{X/Y}$ as $\Delta^* J/J^2$ where $\Delta : X \rightarrow X \times_Y X$ is the diagonal embedding and J/J^2 is the normal bundle of $\Delta(X)$ in $X \times_Y X$. In the case when $Y = \text{Spec}(L)$ for $k \subset L \subset K$ and k the field of definition of X , we use the notation $\Omega_{X/L}$ to denote the sheaf of meromorphic differentials on X and $\Omega_{X/L}^*$ the sheaf of meromorphic vector fields. There is a canonical isomorphism;

$$d : m_z/m_z^2 \rightarrow (\Omega_{X/L})_z \otimes L$$

$$d(f + m_z^2) = df$$

relating the sheaf of differentials to the cotangent space at a point. Using this isomorphism and Nakayama's Lemma, one has that for an algebraic variety X of dimension n over $k \subset L$, $\Omega_{X/L}$ is a locally free module of rank n on the nonsingular locus U of X , see [9] for details.

1. ZARISKI STRUCTURES

Definition 1.1. *Let (\mathcal{M}, τ) be a topological space and let $\{C\}$ denote the collection of closed sets on (\mathcal{M}) . We call (\mathcal{M}, τ) a Zariski structure if the following axioms hold;*

(L) Language: Basic relations are closed;

The diagonals $\Delta_i \subset \mathcal{M}^i \times \mathcal{M}^i$ are closed.

Any singleton in \mathcal{M} is closed.

Cartesian products of closed sets are closed

(P) Properness: The projection maps $pr : \mathcal{M}^{n+1} \rightarrow \mathcal{M}^n$ are proper and continuous, that is the images and inverse images of closed sets under pr are closed

(DCC) Descending Chain Condition: The topology given by the closed sets on \mathcal{M}^n is Noetherian for all $n \geq 1$. The condition (DCC)

implies that every closed set C can be written uniquely (up to permutation) as a union of irreducible closed sets;

$$C = C_1 \cup \dots \cup C_n$$

(DIM) Dimension: The following notion of dimension for closed sets $C \subset \mathcal{M}^n$ is well defined;

For irreducible C , $\dim(C)$ is the maximum m for which there exists a chain of irreducible closed sets $C_0 \subset C_1 \subset \dots \subset C_m = C$.

For arbitrary closed C , $\dim(C) = \max_{1 \leq i \leq m} \{\dim(C_i)\}$ for C_i the irreducible components of C

(PS) Pre-Smoothness: For all closed irreducible sets $C_1, C_2 \subset \mathcal{M}^n$, with $C_1 \cap C_2 \neq \emptyset$,

$$\dim_{\text{comp}}(C_1 \cap C_2) \geq \dim(C_1) + \dim(C_2) - \dim(\mathcal{M}^n)$$

(DF) Definability of fibres: If $C \subset \mathcal{M}^{n+m}$ is closed, then

$$F(C, k) = \{\bar{a} \in \mathcal{M}^n : \dim(C(\bar{a})) > k\}$$

is closed.

(GF) Generic fibres: If $C \subset \mathcal{M}^{n+m}$ is closed and irreducible, then

$$\dim(C) = \dim(\text{pr}(C)) + \min_{\bar{a} \in \text{pr}(C)} \dim C(\bar{a})$$

Remarks 1.2. *The definition of dimension easily implies the following properties;*

(DU) *Dimension of unions: For C_1, C_2 closed, then*

$$\dim(C_1 \cup C_2) = \max\{\dim(C_1), \dim(C_2)\}$$

(DP) *The dimension of a point is 0.*

(DI) *Dimension of irreducible sets: If $C_1 \subsetneq C_2$ and C_2 is irreducible, then $\dim(C_1) < \dim(C_2)$.*

We now show the following;

Theorem 1.3. *Let V be a smooth projective variety of dimension m defined over k and $k \subset L$ with L algebraically closed, then $V(L)$ considered as a topological space with closed sets given by the algebraic subvarieties defined over L is an irreducible Zariski structure of dimension m .*

Proof. We will verify the axioms;

(L) We need only verify that the diagonals $\Delta_i \subset V^i \times V^i$ are closed.

(P) An algebraic variety V is complete if for all varieties Y , the projection morphism

$$pr : V \times Y \rightarrow Y$$

is closed. Taking Y to be V^n in the above definition, complete varieties have the property that the projection maps

$$pr : V^{n+1} \rightarrow V^n$$

are closed. If $W \subset V$ is a closed subvariety of a complete variety V , then, as is easily checked, W is also complete. By assumption V is a closed subvariety of $P^N(L)$ for some N . Now it is a classical fact that $P^N(L)$ is complete, see for example [5].

(DCC) Let $\{W_i : i < \omega\}$ be an infinite descending chain of closed subvarieties of V^n . Let $\{U_1, \dots, U_n\}$ be an affine open cover of V^n . Then $\{U_j \cap W_i : i < \omega\}$ defines a descending chain of closed subvarieties of each U_j . By the Nullstellensatz, each such chain stabilises inside U_j . Then clearly the chain stabilises inside V^n .

(DIM) For W an irreducible subvariety of V^n , we let $\dim_{geom}(W) = t.deg(L(W)/L)$. Then \dim_{geom} corresponds to \dim as defined above. To see this, suppose that $\dim(W) \geq n + 1$, and W is irreducible, then by definition one can find an irreducible closed subvariety $W' \subset W$ with $\dim(W') \geq n$ and so inductively $\dim(W') \geq n$. Now take any affine open subset of V^n intersecting W' , so we may assume that W and W' are affine as the function fields are unchanged. Let $L[W]$ denote the coordinate domain of W , p the proper prime ideal corresponding to W' and \dim_{Krull} the Krull dimension of an integral domain.

By Krull's theorem, $\text{height}(p) + \dim_{\text{Krull}}(L[W]/p) = \dim_{\text{Krull}}(L[W])$, $\dim_{\text{Krull}}(L[W]) = t.\deg(L(W))$ and $\dim_{\text{Krull}}(L[W]/p) = t.\deg(L(W'))$, hence $t.\deg(L(W')) < t.\deg(L(W))$. It follows that $\dim_{\text{geom}}(W') < \dim_{\text{geom}}(W)$ and so $\dim_{\text{geom}}(W) \geq n+1$. Conversely, if $\dim_{\text{geom}}(W) \geq n+1$, then again assuming W is irreducible and affine, if we take $f \in L[W]$ to be a non-unit, then each irreducible component of $V(f) \subset W$ has codimension 1 in X , see [5]. Therefore, $\dim_{\text{geom}}(V(f)) \geq n$ and inductively $\dim(V(f)) \geq n$. As each component of $V(f)$ is a proper closed subset of X , $\dim(W) \geq n+1$. Now clearly we have that \dim_{geom} corresponds to \dim and so in particular we know that $\dim(V^n) = mn$ and the notion of \dim on V^n is well defined.

(PS) A simple calculation shows that for $(x_1 \dots x_n) \in V^n$, $m_{\bar{x}} \cong \sum_{i=1}^n O_{x_1 \dots \hat{x}_i \dots x_n} \otimes m_{x_i}$. Hence,

$$\text{Tan}_{\bar{x}}(V^n) = (m_{\bar{x}}/m_{\bar{x}}^2)^* \cong \sum_{i=1}^n (m_{x_i}/m_{x_i}^2)^* = \sum_{i=1}^n \text{Tan}_{x_i}(V).$$

Therefore, V^n is smooth.

Now we use the following lemma;

Lemma 1.4. *If X is a non-singular algebraic variety of dimension n , and Y, Z are irreducible closed subsets. Then if W is a component of $Y \cap Z$, we have,*

$$\dim(W) \geq \dim(Y) + \dim(Z) - n$$

or equivalently

$$\text{codim}(W) \leq \text{codim}(Y) + \text{codim}(Z)$$

Proof. We have that $Y \cap Z \cong Y \times Z \cap \Delta(X)$ inside $X \times_L X$. Let g_1, \dots, g_n be uniformizers on an open subset U inside X . Then we saw above that $\Omega_{X/L}$ is just the pullback of the conormal sheaf J/J^2 for the inclusion of $\Delta(X)$ inside $X \times_L X$. As $\Omega_{X/L}$ is locally free, so is J/J^2 , and in particular generated freely on $\Delta(U)$ by the functions $g_1 \otimes 1 - 1 \otimes g_1, \dots, g_n \otimes 1 - 1 \otimes g_n$. At a point $x \in \Delta(U)$, we have that $g_1 \otimes 1 - 1 \otimes g_1, \dots, g_n \otimes 1 - 1 \otimes g_n$ generate J_x/J_x^2 and therefore form a basis for the vector space $J_x/m_x J_x$ as clearly any function belonging to J_x lies in m_x the ideal of functions in $O_{X \times X, x}$ vanishing at x . Then, as $J_x/m_x J_x$ is just the base change $J \otimes k(x)$ of the ideal sheaf J at the point x , it follows by Nakayama's lemma that these functions generate

J on an open neighborhood U containing x (not freely!). It follows that $Y \times Z \cap \Delta(X)$ is cut out by exactly n equations inside $Y \times Z$, so by standard dimension theory we have the result.

□

It follows immediately that V^n satisfies (PS) .

In order to check the final 2 axioms we introduce the following definitions;

Definition 1.5. *If $\bar{a}, \bar{b} \in V^n$ are tuples of elements, we define $\text{locus}(\bar{a}/\bar{b})$ to be the intersection of all closed subvarieties defined over $k(\bar{b})$ containing \bar{a} and $\text{locus}_{\text{irr}}(\bar{a}/\bar{b})$ to be the intersection of all closed subvarieties defined over $k(\bar{b})^{\text{alg}}$. We define $\dim(\bar{a}/k)$ to be $t.\deg(k(\bar{a})/k)$ and $\dim(\bar{a}/k\bar{b})$ to be $t.\deg(k(\bar{a})/k(\bar{b}))$; if the underlying field k is clear from context, we will abbreviate this to $\dim(\bar{a}/\bar{b})$*

By the condition DCC , it is clear that locus and $\text{locus}_{\text{irr}}$ are well defined. $\text{locus}_{\text{irr}}$ is an irreducible subvariety of V^n containing \bar{a} , as if V is an irreducible components of $\text{locus}_{\text{irr}}$ containing \bar{a} and k_V is the minimal field of definition, then k_V has only finitely many conjugates under an automorphism fixing $k(\bar{b})^{\text{alg}}$, hence $k_V \subset k(\bar{b})^{\text{alg}}$.

Definition 1.6. *If $W \subset V^n$ is an irreducible closed subvariety, $\bar{a} \in W$, and \bar{b} is a tuple of elements such that $k(\bar{b})$ contains a field of definition for W , then we say that \bar{a} is generic in W over \bar{b} if $\text{locus}(\bar{a}/\bar{b}) = W$.*

Lemma 1.7. *Let $W \subset V^n$ be an irreducible closed subvariety defined over k and \bar{a} generic in W over k . Then $\dim(W) = t.\deg(k(\bar{a})/k) = \dim(\bar{a}/k)$.*

Proof. By the above, $\dim(W) = \dim_{\text{geom}}(W) = t.\deg(k(W)/k)$. By choosing an open affine subvariety of W containing \bar{a} and defined over k , we can assume that W is affine. Now define a map $ev : k[W] \rightarrow k(\bar{a})$ by setting $ev(f) = f(\bar{a})$. ev is injective as if $f(\bar{a}) = 0$, then as f has coefficients in k and \bar{a} is generic in W over k , $f|_W = 0$. Clearly ev extends to a map on $k(W)$ which is an isomorphism.

□

(DF) Let $W \subset V^{n+m}$ be a closed subvariety and pr the projection onto n factors. We can cover $(P^N(L))^{(n+m)}$ with finitely many affines of the form $A^{N(n+m)}$, hence we may assume that W is a closed subvariety

of $A^{N(n+m)}$ and show that $\Gamma(\bar{y}) = \{\bar{a} : \dim(W(\bar{a})) \geq k+1\}$ is closed in $pr(W)$. By additivity of $t.deg$ and the lemma, this occurs iff we can find algebraically independent elements $b_1 \dots b_k b_{k+1} \subset \bar{b} \subset L$ such that $W(\bar{b}\bar{a})$ holds iff

$$\exists_{\sigma(k+2)} \dots \exists_{\sigma(Nm)} W(x_1, \dots, x_{Nm}, \bar{a})$$

has maximal dimension for some permutation $\sigma \in S_{Nm-(k+1)}$. We may write each projection W_σ in the form

$$\bigcap_i F_i(x_1 \dots x_{k+1}, \bar{y}) = 0 \cap \bigcap_j Q_j(x_1 \dots x_{k+1}, \bar{y}) \neq 0$$

where F_i and Q_j are polynomials in the variables $\bar{x}\bar{y}$. Let $\theta_\sigma(\bar{y})$ define the closed set given by the vanishing of all coefficients in the F_i . Then an easy calculation shows that $\Gamma_\sigma(\bar{y}) = \{\bar{y} \in pr(W) : \theta_\sigma(\bar{y})\}$, which is closed.

(GF) We first show the following;

Lemma 1.8. *Let $W \subset V^{n+m}$ be closed and irreducible, defined over k . Then $\bar{a}\bar{b}$ is generic in W over k iff \bar{a} is generic in $pr(W)$ over k and \bar{b} is generic in $W(\bar{a})$ over $k(\bar{a})$.*

Proof. One direction is straightforward, if \bar{a} is not generic in $pr(W)$, then $\bar{a} \in E \subsetneq pr(W)$ and $\bar{a}\bar{b} \in pr^{-1}(E) \subsetneq W$. If \bar{b} is not generic in $W(\bar{a})$, then we can find $X \subsetneq W(\bar{a})$ containing \bar{b} defined over $k(\bar{a})$. As we are working in a product of $P^m(L)$, we can define X by a series of n -homogeneous equations with coefficients in $k(\bar{a})$. Applying Frobenius to these equations, we can in fact assume that the coefficients lie in $k < \bar{a} >$. Now a straightforward exercise in clearing denominators and writing affine equations in homogeneous form shows that we can write X as the fibre $Y(\bar{a})$ for some closed subvariety Y of V^{n+m} . Intersecting with W if necessary gives a proper closed $Y \subsetneq W$ with $\bar{a}\bar{b} \in W$ and defined over k .

For the other direction, suppose that $\bar{a}\bar{b}$ is not generic in W over k , then there exists X defined over k such that $\bar{a}\bar{b} \in X \subsetneq W$. Then $\bar{a} \in pr(X)$ which is also closed and defined over k . Hence, $pr(X) = pr(W)$. As $\bar{b} \in X(\bar{a})$, we have that $\dim(X(\bar{a})) = \dim(W(\bar{a})) = m$. By (DF),

$$X_m = \{\bar{a} \in pr(X) : \dim(X(\bar{a})) = \dim(W(\bar{a})) = m\}$$

is constructible and, by automorphism, can be seen to be defined over k . Hence, as \bar{a} was assumed to be generic, X_m is open inside $pr(X)$. Now, using Lemma 1.7 and the hypotheses on \bar{a}, \bar{b} , $\dim(X) \geq \dim(\bar{a}\bar{b}/k) = \dim(\bar{b}/\bar{a}k) + \dim(\bar{a}/k) = m + \dim(pr(W))$. However, choosing $\bar{a}'\bar{b}'$ generic in W over k , we have that $\dim(W) = \dim(\bar{a}'\bar{b}'/k) = m + \dim(pr(W))$ by the properties of X_k . Hence, $\dim(X) \geq \dim(W)$ contradicting the fact that $X \subsetneq W$ and W was assumed to be irreducible.

□

Using the lemma, we can give an easy proof of (GF) ;

Let $W \subset V^{n+m}$ be closed, irreducible and defined over k . Choose $\bar{a}\bar{b}$ generic in W over k . Then

$$\dim(W) = \dim(\bar{a}\bar{b}/k) = \dim(\bar{b}/\bar{a}k) + \dim(\bar{a}/k) = \dim(pr(W)) + \min_{\bar{a} \in pr(W)} \dim(W(\bar{a})).$$

The last equality follows from the previous lemma and (DF) .

We have therefore checked all the axioms.

□

Definition 1.9. *Given a closed subvariety W of $V^m(L)$ and a closed $F \subset W \times V^m$, all defined over k , we say that F is a cover of W if $pr(F) = W$ and that $\bar{a} \in W$ is regular for the cover if $\dim F(\bar{a}) = \dim F(\bar{a}')$ for \bar{a}' generic in W over k .*

2. SPECIALISATIONS

In order to apply the technique of specialisations, we fix an algebraically closed field L and construct a universal extension K_ω as follows.

Set $L = K_0$. Construct K_{i+1} inductively as follows;

Let $K_i((t_{i+1}))$ be the field of formal Laurent series in the variable t_{i+1} over the algebraically closed field K_i . Define $K_{i+1} = K_i((t_{i+1}))^{alg}$.

Given the tower of algebraically closed fields $L \subset K_1 \subset K_2 \subset \dots \subset K_i \subset \dots$, we set $K_\omega = \bigcup_{i < \omega} K_i$

For all $i < \omega$, K_{i+1} is equipped with a canonical valuation $\overline{v_{i+1}} : K_{i+1} \rightarrow \mathcal{Z}$ defined as follows;

For $f \in K_i((t_{i+1}))$, we set $v_{i+1}(f) = \text{ord}_{i+1}(f)$, where $\text{ord}_{i+1}(f)$ is the minimum n appearing in the Laurent expansion of f . As is shown in [10], $(K_i((t_{i+1})), v_{i+1})$ is the completion of $(K_i(t_{i+1}), v)$, for the canonical valuation v on the function field $K_i(t_{i+1})$. It follows that $K_i((t_{i+1}))$ is a Henselian field with respect to v_{i+1} . By Hensel's lemma, $K_i((t_{i+1}))^{alg}$ is a union $\bigcup_{i < \omega} K_i((t_{i+1}^{1/n}))$ of ramified extensions of $K_i((t_{i+1}))$. The valuation v_{i+1} extends uniquely to the spectral valuation on $K_i((t_{i+1}^{1/n}))$ by the formula;

$$\overline{v_{i+1}}(\alpha) = (1/n)v_{i+1}(N_{K_i((t_{i+1}^{1/n}))/K_i((t))}(\alpha))$$

From the previous section, we have that $P^n(L)$ and $P^n(K_\omega)$ with closed sets given by subvarieties defined over L and K_ω respectively are Zariski structures. We define a specialisation map $\pi_\omega : P^n(K_\omega) \rightarrow P^n(L)$ as follows. First, the maps;

$$\pi_{i+1,m} : P^n(K_i((t_{i+1}^{1/m}))) \rightarrow P^n(K_i)$$

are defined by

$$(f_0 : \dots : f_n) \mapsto (\overline{t^s f_0} : \dots : \overline{t^s f_n})$$

where $s \in \mathcal{Z}$ is chosen such that $\{t^s f_0, \dots, t^s f_n\} \subset O_{v_{i+1}} = K_i[[t_{i+1}^{1/m}]]$ and $v_{i+1}(t_{i+1}^s f_j) = 0$ for some j with $0 \leq j \leq n$. Using the fact that the residue mapping is a homomorphism on $K_i((t_{i+1}^{1/m}))$, this map is clearly well defined. Moreover, the maps $\pi_{i+1,m}$ are compatible for $m \in \mathcal{N}$, in the sense that, given m_1 and m_2 , for $\pi_{i+1,m_1 m_2} : P^n(K_i((t_{i+1}^{1/m_1 m_2}))) \rightarrow P^n(K_i)$, we have that $\pi_{i+1,m_1 m_2}|_{P^n(K_i((t_{i+1}^{1/m_k})))} = \pi_{i+1,m_k}$ for $k \in \{1, 2\}$. Hence, the maps $\pi_{i+1,m}$ naturally define a map

$$\pi_{i+1} : P^n(K_{i+1}) \rightarrow P^n(K_i)$$

$$(f_0 : \dots : f_n) \mapsto \pi_{i+1,m}(f_0 : \dots : f_n)$$

$$\text{where } \{f_0, \dots, f_n\} \subset K_i[[t^{1/m}]]$$

Now, for $M < N$, let $\Pi_{N,M} = \pi_{M+1} \circ \dots \circ \pi_N : P^n(K_N) \rightarrow P^n(K_M)$ and let $\Pi_{M,M} = Id$. Then we have that for $M_1 \leq M_2 \leq M_3$, $\Pi_{M_1,M_2} \circ$

$\Pi_{M_2, M_3} = \Pi_{M_1, M_3}$, hence the maps $\{\Pi_{N, M}\}$ form an inverse system and we set $\Pi_\omega = \bigcup_{M, N} \Pi_{M, N} : P^n(K_\omega) \rightarrow P^n(L)$.

We now show the following lemmas for the pair $(P^n(K_\omega), \Pi_\omega)$.

Lemma 2.1. *Let $V \subset P^n(L)$ be a smooth projective variety defined over L . Then $\Pi_\omega : V(K_\omega) \rightarrow V(L)$ defines a homomorphism of Zariski structures, in the sense that for all closed $W \subset V^m$ defined over L and $\bar{a} \in W(K_\omega)$, we have that $\Pi_\omega(\bar{a}) \in W(L)$.*

Without loss of generality we can take V to be $P^n(L)$ and consider the case $m = 2$. The Segre embedding is defined by;

$$P^n(L) \times P^n(L) \rightarrow P^{n(n+2)}(L)$$

$$((x_0 : \dots : x_n), (y_0 : \dots : y_n)) \mapsto (x_0 y_0 : \dots : x_0 y_n : x_1 y_0 : \dots : x_n y_n)$$

and the following diagram is easily checked to commute:

$$\begin{array}{ccc} P^n(K_{i+1}) \times P^n(K_{i+1}) & \xrightarrow{\text{Segre}} & P^{n(n+2)}(K_{i+1}) \\ \downarrow \pi_{i+1} \times \pi_{i+1} & & \downarrow \pi_{i+1} \\ P^n(K_i) \times P^n(K_i) & \xrightarrow{\text{Segre}} & P^{n(n+2)}(K_i) \end{array}$$

Therefore, it is sufficient to prove that the property holds for $\pi_{i+1} : P^{n(n+2)}(K_{i+1}) \rightarrow P^{n(n+2)}(K_i)$. This is trivial to check using the fact that the residue map on $K_i[[t^{1/m}]]$ is a ring homomorphism fixing K_i .

Now we show that $(P^n(K_\omega), \Pi_\omega)$ has the following universal property;

Lemma 2.2. *Let $L \subset L_m$ be a field extension of transcendence degree m , V a smooth projective variety defined over L and suppose the map $\pi : V(L_m) \rightarrow V(L)$ is given satisfying the conclusion of Lemma 2.1. Then there exists an L -embedding $\alpha_L : L_m \rightarrow K_\omega$ with the property that $\Pi_\omega \circ \alpha_L = \pi$.*

Choose a transcendence basis $\{t_1, \dots, t_m\}$ for L_m over L . We may assume that V is $P^n(L)$ for some L and that $\pi : P^n(L_m) \rightarrow P^n(L)$ is defined as above for some discrete valuation v on L_m with residue field L . Altering (t_1, \dots, t_m) by automorphism if necessary, we may assume

that $v|L(t_1, \dots, t_m)$ is the canonical valuation given by;

$$v_{res} : L(t_1, \dots, t_m) \rightarrow (\mathcal{Z}^m, <)$$

$$v_{res}(t_1^{i_1} \dots t_m^{i_m}) = (i_1 \dots i_m)$$

where $<$ denotes the lexicographic ordering on \mathcal{Z}^m . The completion of $L(t_1, \dots, t_m)$ with respect to v_{res} is the formal Laurent series in m variables $(L((t_1, \dots, t_m)), \overline{v_{res}})$. Let (\hat{L}_m, \bar{v}) be the completion of L_m with respect to v , then, as $L((t_1, \dots, t_m))$ is Henselian with respect to \bar{v}_{res} , \bar{v} is the unique extension of \bar{v}_{res} to \hat{L}_m . Now, for $1 \leq i \leq m$, there exist canonical isomorphisms between $L((t_i, \dots, t_m))$ and $L((t_i)((t_{i+1}, \dots, t_m)))$. These combine to give an L -embedding α_L of $L((t_1, \dots, t_m))$ into K_m . Moreover, an easy calculation shows that $\Pi_\omega \circ \alpha_L = \pi$ on $L(t_1, \dots, t_m)$. Now, by the uniqueness of the valuation extension from $\alpha(L((t_1, \dots, t_m)))$ to $K_m = \alpha(L((t_1, \dots, t_m)))^{alg}$, for any extension of α to an embedding of L_m into K_m , we have $\Pi_\omega \circ \alpha_L = \pi$ on L_m as well.

3. INFINTESIMAL NEIGHBORHOODS

From now on, we fix a pair of Zariski structures and the specialisation map Π_ω , to give a triple $((V(L), V(K_\omega), \Pi_\omega)$ where V is a smooth projective variety defined over L .

Definition 3.1. For $\bar{a} \in V(L)^n$, we define the infinitesimal neighborhood of \bar{a} to be;

$$\mathcal{V}_{\bar{a}} = \Pi_\omega^{-1}(\bar{a})$$

The first property of infinitesimal neighborhoods is that we can move inside closed sets.

Lemma 3.2. If $W(\bar{y})$ is an irreducible closed set defined in $V(L)$, $\bar{b} \in W$ and $\dim(W) = r$, then there exists a $\bar{b}' \in \mathcal{V}_{\bar{b}} \cap W(K_\omega)$ such that $\dim(\bar{b}'/L) = r$

Proof. Consider the collection of constructible sets inside $V(L)^n$

$$W(\bar{y}) \cup \{\neg C(\bar{y}) : C \text{ closed, definable over } L, \dim(W(\bar{y}) \cap C(\bar{y})) < r\}$$

As W is irreducible of dimension r , any finite subcollection has a realisation in $V(L)^n$. By compactness, we can find a realisation \bar{b}' in $W(K)$ for $L \subset K$ such that $\dim(\bar{b}'/L) = r$. It then follows that we can define a partial specialisation $\pi : V(K) \rightarrow V(L)$ by setting $\pi(\bar{b}') = \bar{b}$, for if $C(\bar{y})$ is a closed set defined over L such that $\neg C(\bar{b})$, then we must have that $\dim(W(\bar{y}) \cap C(\bar{y})) < d$ otherwise, W being irreducible, $W(\bar{y}) \subset C(\bar{y})$, so by construction $\neg C(\bar{b}')$ also holds. Now, using Lemma 2.2 applied to the field $L(\bar{b}')$ which has transcendence degree r over L , we may assume that $L(\bar{b}') \subset K_r \subset K_\omega$ and the specialisation π is given by the restriction of Π_ω . \square

We now come to the critical theorem, a more general version of which was originally proved by Zilber in the context of abstract Zariski structures;

Theorem 3.3. *Suppose that $F \subset D \times V^k$ is an irreducible finite cover of D with D a smooth subvariety of V^m , and F, D defined over L , such that $F(a, b)$. If $a' \in \mathcal{V}_a \cap D(K_\omega)$ is generic in D over L , then we can find $b' \in \mathcal{V}_b$ such that $(a', b') \in F(K_\omega)$.*

We here only sketch the proof, full details may be found in [?]. We first consider the following collection of constructible sets defined over K_ω , with $a' \in \mathcal{V}_a \cap D(K_\omega)$ generic over L ;

$$\{F(a', y)\} \cup \{\neg C(d, y) : d \in V(K_\omega), \neg C(\Pi_\omega(d), b)\}$$

As F is a finite cover and K_ω is algebraically closed, a realisation b' of this collection lies in $V(K_\omega)$ and $F(a', b')$ holds. Moreover, $\Pi_\omega(b') = b$, otherwise, as the diagonal $x = y$ is closed, we have that $b' \neq y$ is in the collection which is ridiculous.

If the collection is inconsistent, we find a closed set $Q \subset V^{n+k}$ such that $F(a', y) \subseteq Q(d, y)$ whereas $\neg Q(\pi(d), b)$.

The point of the smoothness assumption is to show that the parameter space

$$L(x, z) \subset D \times V^n = \{(x, z) : F(x, y) \subset Q(z, y)\}$$

which in general is not relatively closed in $D \times M^n$ at least corresponds to a closed set over a dense open subset of D . More precisely, there is a closed subvariety $P(x, z) \subset D \times M^n$ and $D' \subset D$, $\dim(D') < \dim(D)$, all defined over L , such that

1. $P(x, z) \subset L(x, z)$.
2. $L(x, z) \subset P(x, z) \cup (D' \times V^n) \quad (*)$

We have by assumption that $L(a', d)$ holds. As a' was chosen to be generic over L and $a' \notin D'$, $P(a', d)$ holds. Applying the specialisation Π_ω gives that $P(a, \Pi_\omega(d))$, hence $F(a, y) \subset Q(\Pi_\omega(d), y)$, hence $Q(\Pi_\omega(d), b)$ holds as well, contradicting the assumption.

Remarks 3.4. *In fact the theorem can be improved to give the following more general result;*

Suppose that $F \subset D \times V^k$ is an irreducible generically finite cover of D with D a subvariety of V^m . Then, if $a \in D$ is a regular point for the cover and contained in the non-singular locus of D , $a' \in \mathcal{V}_a \cap D(K_\omega)$ is generic in D over L , then we can find $b' \in \mathcal{V}_b$ such that $(a', b') \in F(K_\omega)$.

4. ZARISKI UNRAMIFIED MAPS AND MULTIPLICITY

The purpose of introducing infinitesimal neighborhoods is to define an abstract notion of Zariski multiplicity.

Definition 4.1. *Zariski multiplicity*

Let hypotheses be as in Theorem 3.3

Given $(a, b) \in F$, set

$$\text{mult}_{ab}(F/D) = \text{Card}(F(a', K_\omega) \cap \mathcal{V}_b) \text{ for } a' \in \mathcal{V}_a \cap D \text{ generic over } L$$

We want to show this is well defined.

Proof. Suppose $a'' \in \mathcal{V}_a \cap D$ with $\text{Card}(F(a'', K_\omega) \cap \mathcal{V}_b) = n$. Consider the relation $N(x, y_1, \dots, y_n) \subset D \times V^{nk}$, given by

$$N(x, y_1, \dots, y_n) = F(x, y_1) \wedge \dots \wedge F(x, y_n)$$

Then we have that N is a finite cover of D and, moreover, by smoothness of D , each irreducible component of N has dimension at least

$$n(\dim(F) + (n-1)k) - (n-1)(\dim(D) + nk) = \dim(D) + n(n-1)k - n(n-1)k = \dim(D)$$

so clearly each component is a finite cover of D . Now, choose an irreducible component N_i containing $(a'', b_1'', \dots, b_n'')$, so by specialisation also contains (a, b, \dots, b) and consider the open set $U \subset N_i$ given by

$$U(x, y_1, \dots, y_n) = N_i(x, y_1, \dots, y_n) \wedge y_1 \neq y_2 \neq \dots \neq y_n$$

Then, for $a' \in \mathcal{V}_a$ generic in D , it follows we can find a tuple (b'_1, \dots, b'_n) such that $N_i(a', b'_1, \dots, b'_n)$, and $(b'_1, \dots, b'_n) \in \mathcal{V}_{(b, \dots, b)}$. As is easily checked, the tuple (a', b'_1, \dots, b'_n) is generic inside N_i , hence must lie inside U . This proves that the b'_1, \dots, b'_n are distinct, hence $\text{Card}(F(a', K_\omega) \cap \mathcal{V}_b) \geq n$. □

Definition 4.2. We say that a point $(ab) \in F$ is Zariski ramified if $\text{mult}_{ab}(F/D) \geq 2$. Otherwise, we call such a point Zariski unramified.

Now suppose $F \subset D \times V^n$ is an irreducible finite cover of D with D smooth, then we have the following easily checked lemma

Lemma 4.3. $\text{mult}_a(F/D) =_{\text{def}} \sum_{b \in F(a, L)} \text{mult}_{ab}(F/D)$ does not depend on the choice of $a \in D$, and is equal to the size of a generic fibre over D

A simple consequence is the following:

Lemma 4.4. If $\bar{a}' \in D(L)$, then $F(\bar{a}')$ contains a point of ramification in the sense of Zariski structures iff $|F(\bar{a}')| < |F(\bar{a})|$ where \bar{a} is generic in D .

Proof. We have seen that $|F(\bar{a})| = \sum_{\bar{b} \in F(\bar{a}', L)} \text{mult}_{\bar{a}', \bar{b}}(F/D)$. If $|F(\bar{a}')| < |F(\bar{a})|$, then there must exist $\bar{b} \in F(\bar{a}')$ with $\text{mult}_{(\bar{a}', \bar{b})}(F/D) \geq 2$ so the result follows by the definition of ramification in Zariski structures. The converse is similar. □

We will also require the following results, that Zariski multiplicity is multiplicative over composition and preserved by open maps.

Lemma 4.5. *Suppose that F_1, F_2 and F_3 are smooth, irreducible, with $F_2 \subset F_1 \times V^k$ and $F_3 \subset F_2 \times V^l$ finite covers. Let $(abc) \in F_3 \subset F_1 \times V^k \times V^l$. Then $\text{mult}_{abc}(F_3/F_1) = \text{mult}_{ab}(F_2/F_1)\text{mult}_{bc}(F_3/F_2)$.*

Proof. To see this, let $m = \text{mult}_{ab}(F_2/F_1)$ and $n = \text{mult}_{bc}(F_3/F_2)$. Choose $a' \in \mathcal{V}_a \cap F_1(K_\omega)$ generic over L . By definition, we can find distinct $b_1 \dots b_m$ in $V^k(K_\omega) \cap \mathcal{V}_b$ such that $F_2(a', b_i)$ holds. As F_2 is a finite cover of F_1 , we have that $\dim(a'b_i/L) = \dim(a'/L) = \dim(F_1) = \dim(F_2)$, so each $(a'b_i) \in \mathcal{V}_{ab} \cap F_2$ is generic over L . Again by definition, we can find distinct $c_{i1} \dots c_{in}$ in $V^l(K_\omega) \cap \mathcal{V}_c$ such that $F_3(a'b_i c_{ij})$ holds. Then the mn distinct elements $(a'b_i c_{ij})$ are in \mathcal{V}_{abc} , so by definition of multiplicity $\text{mult}_{abc}(F_3/F_1) = mn$ as required. \square

Lemma 4.6. *Let $\pi_1 : X \rightarrow D$ and $\pi_2 : Y \rightarrow D$ be covers with assumptions as in remarks following Theorem 3.3. Suppose moreover that there exist open smooth subvarieties $U \subset X$ and $V \subset Y$ and an isomorphism $f : U \rightarrow V$ such that $\pi_2 \circ f = \pi_1$ on U . Then if $a \in D$ is a regular point for the cover π_1 and $(ab) \in U$, $\text{mult}_{ab}(X/D) = \text{mult}_{af(b)}(Y/D)$.*

Proof. We may assume that the open set U is maximal with the property that $(ab) \in U$ and there exists an isomorphism with $V \subset Y$. Suppose $\text{mult}_{ab}(X/D) = m$. Then we can find $a' \in \mathcal{V}_a \cap D(K_\omega)$ generic in D over L and b_1, \dots, b_m distinct such that $X(a'b_i)$ holds for $1 \leq i \leq m$. It will be sufficient to show that $Y(a'f(b_i))$ holds and $f(b_i) \in \mathcal{V}_{f(b)}$, for $1 \leq i \leq m$, then, as f is injective, $\text{mult}_{a,f(b)}(Y/D) \geq m$ and the result follows by symmetry. By the fact that $\pi_2 \circ f = \pi_1$ on U we clearly have that $Y(a'f(b_i))$ holds. Let $\overline{\text{graph}(f)}$ be the projective closure of the graph of f in the projective variety $X \times Y$ and π_X, π_Y the projections onto the coordinates X and Y . Then π_X satisfies the conditions of the remarks after Theorem 3.3, and moreover by assumption the point $(ab) \in X$ is regular for the cover π and contained in the non-singular locus of X . Hence, we can find $(cd) \in \mathcal{V}_{af(b)}$ such that $\overline{\text{graph}(f)}(a'b_i, cd)$ holds. As $\text{graph}(f)$ is a 1–1 correspondence between U and V , if $(a'b_i, cd) \in \overline{\text{graph}(f)} \setminus \text{graph}(f)$ then $(a'b_i, cd) \in F_X \cup F_Y$ where F_X, F_Y consist of the infinite fibres of the projections π_X and π_Y respectively. By (DF) , both of these are defined over L and have dimension strictly less than $\overline{\text{graph}(f)}$. This contradicts the fact that $(a'b_i, cd)$ is generic inside $\overline{\text{graph}(f)}$ over L , hence $(a'b_i, cd) \in \text{graph}(f)$ and as f is a bijection $(cd) = (a'f(b_i))$. This shows that $f(b_i) \in \mathcal{V}_{f(b)}$ as required. \square

5. ETALE MORPHISMS AND ALGEBRAIC MULTIPLICITY

We review here the algebraic notions which will be required in the following section.

Definition 5.1. *A morphism f of finite type between varieties X and Y is said to be etale if for all $x \in X$ there are open affine neighborhoods U of x and V of $f(x)$ with $f(V) \subset U$ such that restricted to these neighborhoods the pull back on functions is given by the inclusion;*

$$f^* : L[V] \rightarrow L[V] \frac{[x_1, \dots, x_n]}{f_1, \dots, f_n}$$

$$\text{and } \det\left(\frac{\partial f_i}{\partial x_j}\right)(x) \neq 0, (*)$$

The coordinate free definition of etale is that f should be flat and unramified.

The notion of an etale morphism simplifies considerably when we assume that X and Y are smooth algebraic varieties over L , see [9];

Theorem 5.2. *If X and Y are non-singular algebraic varieties over L and $f : X \rightarrow Y$ is a morphism, then f is etale iff $df : (m_x/m_x^2)^* \rightarrow (m_{f(x)}/m_{f(x)}^2)^*$ is an isomorphism everywhere.*

Remarks 5.3. *This gives us a convenient test for etaleness given an arbitrary morphism of finite type between smooth varieties X and Y . If we take local uniformisers g_1, \dots, g_n at $x \in X$, the dg_i generate Ω_X freely on an open U' of x . If we pull back a set of uniformisers f^*f_1, \dots, f^*f_n on Y to X , we can locally define the Jacobian Jac_g^f to be;*

$$\det\left(\frac{\partial f^*f_i}{\partial g_j}\right)$$

*which means write the 1-forms $f^*df_i = \sum_j a_{ij}dg_j$ and take $\det(a_{ij})$. If f is etale in a neighborhood of x , the f^*df_i also generate Ω_X freely on an open U'' of x . Taking the intersection $U'' = U \cap U'$, gives us that the Jacobian $Jac_g^f|_{U''} \neq 0$. Conversely, if $Jac_g^f(x) \neq 0$, then it is non zero on an open neighborhood U'' of x and by the above theorem we have that f is etale on this neighborhood.*

We will also require some facts about the etale topology on an algebraic variety Y . We consider a category Y_{et} whose objects are etale morphisms $U \rightarrow Y$ and whose arrows are Y -morphisms from $U \rightarrow V$. This category has the following 2 desirable properties. First given

$y \in Y$, the set of objects of the form $(U, x) \rightarrow (Y, y)$ form a directed system, namely $(U, x) \subset (U', x')$ if there exists a morphism $U \rightarrow U'$ taking x to x' . Secondly, we can take “intersections” of open sets U_i and U_j by considering $U_{ij} = U_i \times_Y U_j$; the projection maps are easily show to be etale and the composition of etale maps is etale, so $U_{ij} \rightarrow Y$ still lies in Y_{et} . (Note that we can develop the theory of etale cohomology for an arbitrary Zariski structure, this will be a subject of further investigation) If Y is an irreducible variety over K , then all etale morphisms into Y must come from reduced schemes of finite type over K , though they may well fail to be irreducible considered as algebraic varieties. Now we can define the local ring of Y in the etale topology to be;

$$O_{y,Y}^\wedge = \lim_{\rightarrow, y \in U} O_U(U)$$

As any open set U of Y clearly induces an etale morphism $U \rightarrow_i Y$ of inclusion, we have that $O_{y,Y} \subset O_{y,Y}^\wedge$. We want to prove that $O_{y,Y}^\wedge$ is a Henselian ring and in fact the smallest Henselian ring containing $O_{y,Y}$. We need the following lemma about Henselian rings;

Lemma 5.4. *Let R be a local ring with residue field k . Suppose that R satisfies the following condition;*

If $f_1, \dots, f_n \in R[x_1, \dots, x_n]$ and $\bar{f}_1 \dots \bar{f}_n$ have a common root \bar{a} in k^n , for which $Jac(\bar{f})(\bar{a}) = (\frac{\partial \bar{f}_i}{\partial x_j})_{ij}(\bar{a}) \neq 0$, then \bar{a} lifts to a common root in R^n ().*

Then R is Henselian.

It remains to show that $O_{y,Y}^\wedge$ satisfies (*).

Proof. Given f_1, \dots, f_n satisfying the condition of (*), we can assume the coefficients of the f_i belong to $O_{U_i}(U_i)$ for covers $U_i \rightarrow Y$; taking the intersection $U_{1\dots i\dots n}$ we may even assume the coefficients define functions on a single etale cover U of Y . By the remarks above we can consider U as an algebraic variety over K , and even an affine algebraic variety after taking the corresponding inclusion. We then consider the variety $V \subset U \times A^n$ defined by $\text{Spec}(\frac{R(U)[x_1, \dots, x_n]}{f_1, \dots, f_n})$. Letting $u \in U$ denote the point in U lying over $y \in Y$, the residue of the coefficients of the f_i at u corresponds to the residue in the local ring R , which tells us exactly that the point (u, \bar{a}) lies in V . By the Jacobian condition, we have that the projection $\pi : V \rightarrow U$ is etale at the point (u, \bar{a}) , and

hence on some open neighborhood of (u, \bar{a}) , using Nakayama's Lemma applied to $\Omega_{V/U}$. Therefore, replacing V by the open subset $U' \subset V$ gives an étale cover of U and therefore of Y , lying over y . Now clearly the coordinate functions x_1, \dots, x_n restricted to U' lie in $O_{y,Y}^\wedge$ and lift the root \bar{a} to a root in $O_{y,Y}^\wedge$. \square

We define the Henselization of a local ring R to be the smallest Henselian ring $R' \supset R$, with $R' \subset \text{Frac}(R)^{\text{alg}}$. We have in fact that;

Theorem 5.5. *Given an algebraic variety Y , $O_{y,Y}^\wedge$ is the Henselization of $O_{y,Y}$*

Definition 5.6. *Given smooth projective curves C_1, C_2 and a finite morphism $f : C_1 \rightarrow C_2$, the algebraic multiplicity of f at a is $\text{ord}_a(f^*h)$ where h is a local uniformiser for C_2 at $f(a)$.*

Remarks 5.7. *This is independent of the choice of h , as the quotient of 2 uniformisers h/h' is a unit in $\mathcal{O}_{f(a)}$. Given finite morphisms $f : C_3 \rightarrow C_2$ and $g : C_2 \rightarrow C_1$, if $\text{ord}_{a,f(a)}(C_3/C_2) = m$ and $\text{ord}_{f(a),gf(a)}(C_2/C_1) = n$, then taking a local uniformiser h at $gf(a)$, we have that $g^*h = h_1^n u$ locally at $f(a)$ for a unit u and uniformiser h_1 in $\mathcal{O}_{f(a)}$. Similarly $f^*g^*h = h_2^{mn} u'$ for a unit u' and uniformiser h_2 in \mathcal{O}_a . This shows that $\text{ord}_{a,gf(a)}(C_3/C_1) = mn$, so the branching number is also multiplicative for smooth projective curves.*

Definition 5.8. *Given smooth projective varieties X_1, X_2 and a finite morphism $f : X_1 \rightarrow X_2$, the algebraic multiplicity $\text{mult}_{af(a)}^{\text{alg}}(X_1/X_2)$ of f at $a \in X_1$ is $\text{length}(O_{a,X_1}/f^*m_{f(a)})$ where $m_{f(a)}$ is the maximal ideal of the local ring $O_{f(a)}$.*

Remarks 5.9. *Note that this is finite, by the fact that finite morphisms have finite fibres and the ring $O_{a,X_1}/f^*m_{f(a)}$ is a localisation of the fibre $f^{-1}(f(a)) \cong R(f^{-1}(U)) \otimes_{R(U)} L \cong R(f^{-1}(U))/m_{f(a)}$ where U is an affine subset of X_2 containing $f(a)$.*

We now have the following, which generalises the result for curves;

Theorem 5.10. *Algebraic multiplicity is multiplicative;*

Given finite morphisms $f : X_3 \rightarrow X_2$ and $g : X_2 \rightarrow X_1$ between smooth projective varieties, for $a \in X_3$ we have that

$$\text{mult}_{af(a)}(X_3/X_2)\text{mult}_{f(a)gf(a)}(X_2/X_1) = \text{mult}_{agf(a)}(X_3/X_1).$$

Proof. The proof is an exercise in algebra, which we give for want of a convenient reference. First, the morphisms f and g are flat. This requires the following lemma, given as an exercise in [4], and the fact that smooth varieties are regular and Cohen-Macaulay;

Lemma 5.11. *Let $f : X \rightarrow Y$ be a morphism of varieties over L . Assume that Y is regular, X is Cohen-Macaulay and that every fibre of f has dimension equal to $\dim(X) - \dim(Y)$. Then f is flat.*

Now we have a tower of local rings $(R, m) \subset (S, n) \subset (T, o)$ with algebraically closed residue field L . Each extension is free by the flatness result and finiteness. For a finite free extension $(R, m) \subset (S, n)$ of local rings, we also have the easily checked result that;

$$[S : R] = \dim_{Fr(R)} S \otimes_R Fr(R) = \dim_L S \otimes_{R/mR} R/mR = \dim_L (S/mS)(*).$$

For an extension $(R, m) \subset (S, n)$ of local rings, we have that $\text{length}(S/mS) = \dim_L(S/mS)$, hence, by (*), the theorem reduces to checking that $[T : R] = [T : S][S : R]$ which is standard.

□

6. EQUIVALENCE OF THE NOTIONS

This section is devoted to the main proofs of the paper, namely that the notions developed in Sections 4 and 5 are essentially equivalent for morphisms between smooth projective varieties.

Theorem 6.1. *Let hypotheses be as in Theorem 3.3, with the additional assumption that $\text{char } L = 0$, then F is a Zariski unramified cover of D iff F is an étale cover of D .*

Let pr be the projection map of F onto D , then pr is a projective morphism. By Zariski's Main Theorem, pr factors as a composition $F \xrightarrow{pr_1} F' \xrightarrow{pr_2} D$ with pr_1 having connected fibres, $pr_{1*}F = F'$ and pr_2 a finite morphism. The formal inverse pr_1^{-1} from F' to F is a morphism corresponding to the identification of $pr_{1*}F$ and F' , hence pr_1 is in fact an isomorphism. We may therefore assume that pr is a finite morphism.

Now suppose that pr is etale, then, pr is flat, see [9] for how this follows from Definition 5.1. As D is irreducible,

$$\dim_{k(y)}(f_*(O_F) \otimes_{O_y} k(y))$$

is independent of $y \in D$. As pr is etale, $pr_* : T_{x,F} \rightarrow T_{pr(x),D}$ is an isomorphism, hence, by a simple calculation;

$$\dim_{k(y)}(f_*(O_F) \otimes_{O_y} k(y)) = |F(y)| \text{ for } y \in D.$$

This shows that $|F(y)|$ is independent of $y \in D$. By Lemma 4.4, this shows that pr is a Zariski unramified cover.

Conversely, suppose that pr is Zariski unramified. We first show that for generic $\bar{a} \in D$, $|F(\bar{a})| = \deg(pr) = \deg[k(F) : k(D)]$. As $\text{char}(k(F)) = 0$, the extension is separable so we can find a primitive element $g \in k(F)$ such that $k(F) = k(D)(g)$. Clearly the minimum polynomial p of g over $k(D)$ has degree $n = \deg[k(F) : k(D)]$. Let $h_1, \dots, h_{n-1} \in k(D)$ be the coefficients of p , then $R(D)(h_1 \dots h_{n-1})$ determines the function ring of a Zariski open subset U of D . Clearly $R(U)[g]$ is an integral extension of $R(U)$ and corresponds to the projection restricted to $U' = pr^{-1}(U) \cap g \neq 0$. By dimension theory, the zero set $Z(g) \subset D$ cannot intersect with a generic fibre of the original map $pr : F \rightarrow D$. Now we consider the discriminant $D(p)$ of the polynomial p as a regular function on U and we have that for generic $\bar{a} \in U$ that $D(p)(\bar{a}) \neq 0$. This implies that for generic $\bar{a} \in U$ $|pr^{-1}(\bar{a})| = n = \deg[k(F) : k(D)]$. Now we are in a position to apply Theorem 5, p145, of [13] which requires that D should be smooth, namely that $pr_* : T_{x,F} \rightarrow T_{pr(x),D}$ is an isomorphism for $x \in F$. As F and D were assumed to be nonsingular, this is sufficient to show that pr is etale by Theorem 5.2.

Remarks 6.2. When $\text{char}(L) = p$, the analogy fails. If we consider the Frobenius map $Fr : P^1 \rightarrow P^1$, then $\text{Graph}(Fr) \subset P^1 \times P^1$ is a finite cover of P^1 and both $\text{Graph}(Fr)$ and P^1 are smooth. The projection map pr onto the second coordinate is unramified in the sense of Zariski structures as pr is a bijection. However pr fails to be etale in the sense of algebraic geometry as $pr_* : T_{x,\text{Graph}(Fr)} \rightarrow T_{pr(x),P^1}$ is zero everywhere. However the following theorem shows that this is the only bad example and highlights one advantage of the Zariski method, namely that it is insensitive only to Frobenius.

Theorem 6.3. *Let hypotheses be as in Theorem 3.3, with the additional assumption that $\text{char}(L) = p \neq 0$. If F is an etale cover of D , F is a Zariski unramified cover. Conversely, if F is a Zariski unramified cover, then pr factors as a composition $F \rightarrow_{pr_1} F' \rightarrow_{pr_2} D$ in Proj with pr_1 a purely inseparable connected cover and pr_2 an etale cover.*

Proof. As in the previous theorem, we may assume that pr is a finite morphism. Suppose first that $F \rightarrow D$ is a finite morphism with F and D affine. We first find a field L such that $k(F)/L$ is a purely inseparable extension and $L/k(D)$ is separable. Let R' be the integral closure of $R(D)$ in L and R'' the integral closure of $R(D)$ in $k(F)$. As $R(F)$ is integral over $R(D)$ we have that $R(F) \subset R''$, but F was assumed to be smooth so $R(F)$ is integrally closed in $k(F)$ and therefore $R'' = R(F)$. As the extensions $k(D) \subset L \subset k(F)$ are finite algebraic, by [14], both $R(F)$ and R' are finite R' and $R(D)$ modules respectively. Therefore, corresponding to the ring inclusions

$$R(D) \rightarrow R' \rightarrow R(F)$$

we have the sequence of finite morphisms

$$F \rightarrow_{pr_1} \text{Spec}(R') \rightarrow_{pr_2} D$$

We first consider the cover $F \rightarrow_{pr_1} \text{Spec}(R')$. Let g_1, \dots, g_m generate $R(F)$ over R' . As the extension $k(F)/L$ is purely inseparable, we can write the minimum polynomials p_i of g_i in the form $r_{i,0}g^{p^{n_i}} - r_{i,1} = 0$ where $r_{i,0}$ and $r_{i,1}$ are in R' . As $R(F)/R'$ is finite, we can also find monic polynomials q_i with coefficients in R' satisfied by g_i . Choose polynomials $t_i = s_{i,0}x^{m_i} + s_{i,1}x^{m_i-1} + \dots + s_{i,m_i}$ such that $p_i t_i = q_i$. By equating coefficients, we have that $r_{i,0} = s_{i,0}^{-1}$ and $r_{i,1}/r_{i,0} \in R'$. Hence, we can take the p_i to be monic with coefficients in R' . As the p_i are minimal monic polynomials, we conclude that $R(F)$ is an extension of the form $R'[g_1, \dots, g_m]/(g_1^{p^{n_1}} - \lambda_1, \dots, g_m^{p^{n_m}} - \lambda_m)$ with $\lambda_i \in R'$. This is easily checked to be a connected cover of $\text{Spec}(R')$. In fact if we let $\theta = (Fr^{-n_1}, \dots, Fr^{-n_m}) \circ (\lambda_1 \dots \lambda_m)$, where the λ_i are considered as regular functions on $\text{Spec}(R')$ and Fr^{-n_i} is the formal inverse Frobenius map, then the cover corresponds to the projection of $\text{Graph}(\theta) \subset \text{Spec}(R') \times A^m$ onto $\text{Spec}(R')$. As F was assumed to be smooth, $\text{Spec}(R')$ is a smooth separable Zariski unramified cover of D . Applying the previous theorem, we conclude that $\text{Spec}(R')$ is an etale cover of D . Now, for the case when F and D are projective varieties, let U_i be an affine cover of D and $R'(U_i)$ the corresponding

normalisations. By uniqueness of integral closure, the $R'(U_i)$ patch to form a cover F' of D . In fact, by a classical result, see [9], we may assume that F' is a smooth projective variety. As etaleness is a local condition for smooth varieties, the cover F' is etale. Finally, check that the local maps $pr_1 : F_i \rightarrow R'(U_i)$ patch on overlaps to give a morphism $pr_1 : F \rightarrow F'$. Clearly, this is an inseparable connected cover, in fact if F' is defined by the homogenous equations $\langle f_1, \dots, f_n \rangle$ inside P^N , then F is isomorphic to the closed subvariety of $P^N \times P^m$ defined by the extra equations $\langle Y_i^{p^{n_i}} X_N^{j(i)} - \lambda_i(X_0, \dots, X_N) Y_0^{p^{n_i}} \rangle$ where $1 \leq i \leq m$ and $j(i)$ is the degree of the polynomial λ_i in the affine coordinates $P^N(L)_i$. \square

Remarks 6.4. *We now show that the notions of Zariski multiplicity and algebraic multiplicity coincide when $\text{char}(L) = 0$, as usual with assumptions being as in Theorem 3.3., and find an analogous result when $\text{char}(L) = p$. Unfortunately, it does not seem possible to achieve this by counting points in the fibres, as in the previous theorems, so we need to find a local method. This will be the subject of the remainder of this section.*

For ease of exposition, we first consider the case when F and D are curves. We will point out the necessary modifications for the case when F and D are arbitrary smooth projective varieties in the next theorem.

Theorem 6.5. *Let hypotheses be as in Theorem 3.3, with the additional assumption that $\text{char}(L) = 0$ and F, D are curves. Then the notions of Zariski multiplicity and algebraic multiplicity coincide.*

Proof. As D has a non-constant meromorphic function, we can write D as a finite cover of $P^1(L)$. As we have checked both algebraic multiplicity and Zariski multiplicity are multiplicative over composition, a straightforward calculation shows that we need only check the notions agree for the branched finite cover $\pi : F \rightarrow P^1(L)$. (1)

Now consider this cover restricted to A^1 , let x be the canonical coordinate with $\text{ord}_a(\pi^*(x)) = m$, so we have that $\pi^*x = h^m u$, for u a unit in \mathcal{O}_a and h a uniformiser at a . (2)

As u is a unit and $\text{char}(L) = 0$, the equation $z^m = u$ splits in the residue field of \mathcal{O}_a^\wedge . By Hensel's Lemma and Theorem 5.5, it is solvable in \mathcal{O}_a^\wedge . By the definition of \mathcal{O}_a^\wedge , we can find an etale morphism

$\pi : (U, b) \rightarrow (F, a)$ containing such a solution in the local ring \mathcal{O}_b . We may assume that U is irreducible and moreover, as π is etale, that U is smooth. (3)

Now we can embed U in a projective smooth curve F' and, as F is smooth, extend the morphism π to a projective morphism from F' to F . (4)

We claim that $(ba) \in \text{graph}(\pi) \subset F' \times F$ is unramified in the sense of Zariski structures. For this we need the following fact whose algebraic proof relies on the fact that etale morphisms are flat, see [7];

Fact 6.6. *Any etale morphism can be locally presented in the form*

$$\begin{array}{ccc} V & \xrightarrow{g} & \text{Spec}((A[T]/f(T))_d) \\ \downarrow \pi & & \downarrow \pi' \\ U & \xrightarrow{h} & \text{Spec}(A) \end{array}$$

where $f(T)$ is a monic polynomial in $A[T]$, $f'(T)$ is invertible in $(A[T]/f(T))_d$ and g, h are isomorphisms. (5)

Using Lemma 4.6 and the fact that the open set V is smooth, we may safely replace $\text{graph}(\pi)$ by $\overline{\text{graph}(\pi')} \subset F'' \times F$ where F'' is the projective closure of $\text{Spec}((A[T]/f(T))_d)$, F is the projective closure of $\text{Spec}(A)$ and $\overline{\text{graph}(\pi')}$ is the projective closure of $\text{graph}(\pi')$ and show that $(g(b)a)$ is Zariski unramified. Note that over the open subset $U = \text{Spec}(A) \subset F$, $\overline{\text{graph}(\pi')} = \text{Spec}((A[T]/f(T))_d)$ as this is closed in $U \times F''$. For ease of notation, we replace $(g(b)a)$ by (ba) . (6)

Suppose that f has degree n . Let $\sigma_1 \dots \sigma_n$ be the elementary symmetric functions in n variables T_1, \dots, T_n . Consider the equations

$$\sigma_1(T_1, \dots, T_n) = a_1$$

...

$$\sigma_n(T_1, \dots, T_n) = a_n \quad (*)$$

where a_1, \dots, a_n are the coefficients of f with appropriate sign. These cut out a closed subscheme $C \subset \text{Spec}(A[T_1 \dots T_N])$. Suppose $(ba) \in \text{graph}(\pi') = \text{Spec}(A[T]/f(T))$ is ramified in the sense of Zariski structures, then I can find $(a'b_1b_2) \in \mathcal{V}_{abb}$ with $(a'b_1), (a'b_2) \in \text{Spec}(A(T)/f(T))$ and b_1, b_2 distinct. Then complete (b_1b_2) to an n -tuple $(b_1b_2c'_1 \dots c'_{n-2})$ corresponding to the roots of f over a' . The tuple $(a'b_1b_2c'_1 \dots c'_{n-2})$ satisfies C , hence so does the specialisation $(abbc_1 \dots c_{n-2})$. Then the tuple $(bbc_1 \dots c_{n-2})$ satisfies $(*)$ with the coefficients evaluated at a . However such a solution is unique up to permutation and corresponds to the roots of f over a . This shows that f has a double root at (ab) and therefore $f'(T)|_{ab} = 0$. As (ab) lies inside $\text{Spec}(A[T]/f(T))_d$, this contradicts the fact that f' is invertible in $A[T]/f(T))_d$. (7)

In (2) we may therefore assume that $\pi^*x = h^m$ for h a local uniformiser at a . Now we have the sequence of ring inclusions given by

$$L[x] \rightarrow L[x, y]/(y^m - x) \rightarrow R$$

$$x \mapsto \pi^*x, y \mapsto h$$

where R is the coordinate ring of F in some affine neighborhood of a . It follows that we can factor our original map such that F is etale near a over the projective closure of $y^m - x = 0$. (8)

Again, repeating the argument from (4) to (7), we just need to check that the projective closure of $y^m - x$ has multiplicity m at 0 considered as a cover of $P^1(\bar{k})$. This is trivial, let $\epsilon \in \mathcal{V}_0$ be generic over \mathcal{M} , then as we are working in characteristic 0 we can find distinct $\epsilon_1, \dots, \epsilon_m$ in \mathcal{M}_* solving $y^m = \epsilon$. By specialisation, each $\epsilon_i \in \mathcal{V}_0$. (9) \square

Theorem 6.7. *Let hypotheses be as in Theorem 6.5, with the modification that $\text{char}(L) = p \neq 0$. If e denotes the Zariski multiplicity and d the algebraic multiplicity at $a \in F$, then $d = ep^n$ and π factors as $F \rightarrow_h F' \rightarrow_g D$ with $h = \text{Frob}^n$ and g having algebraic multiplicity e at $h(a)$.*

By Theorem 6.3, we can factor π into a purely inseparable morphism $h : F \rightarrow F'$ and a separable morphism $g : F' \rightarrow D$ with F' a smooth projective curve. Theorem 6.3 shows that h is an integer power of Frobenius and Theorem 6.5 shows that the notions of Zariski multiplicity and algebraic multiplicity coincide for the morphism g . Now

the result follows by the fact that h has algebraic multiplicity p^n everywhere but is Zariski unramified.

Theorem 6.8. *Let hypotheses be as in Theorem 6.5, with the modification that F and D are arbitrary smooth projective varieties. Then the notions of Zariski multiplicity and algebraic multiplicity coincide.*

We will make the necessary modifications to Theorem 6.5;

(1). We use the following classical fact (Projective Normalisation), see [9].

Fact 6.9. *Let $D \subset P^n(L)$ be an r dimensional projective variety. Then there exist $(r+1)$ linear forms $l_0(X), \dots, l_r(X)$ with coefficients in L such that the hyperplane H defined by $l_0 = \dots = l_r = 0$ is disjoint from D . If $\tau : P^n(L) - H \rightarrow P^r(L)$ denotes the projection, then the restriction of τ to D is a finite surjective morphism.*

Now combining this with the result in Section 5 that algebraic multiplicity is multiplicative for morphisms between smooth projective varieties, we need only consider the branched finite cover $\pi : F \rightarrow P^r(L)$

(2). In this case, there is no straightforward way to present the pullbacks of the local uniformisers x_1, \dots, x_r at $\bar{0} \in A^r$. Instead, we have the inclusion $\pi^*O_{\bar{0}, A^r} \subset O_{a, F}$ induced by the map π . Passing to the Henselisations, gives an inclusion $O_{\bar{0}, A^r}^\wedge \subset O_{a, F}^\wedge$ for the etale topology. In the case when π fails to be etale in an open neighborhood of a , this is in fact a proper inclusion. Now choose uniformisers w_1, \dots, w_n for $O_{a, F}$. As a and $\bar{0}$ are smooth points, the completions of the local rings $O_{\bar{0}, A^n}$ and $O_{a, F}$ with respect to the order valuations at a and $\bar{0}$ are isomorphic to the formal power series rings $L[[w_1, \dots, w_n]]$ and $L[[x_1, \dots, x_n]]$ respectively. The following is a classical result used in the proof of the Artin approximation theorem, relating the Henselisation of the ring $L\{x_1, \dots, x_n\}$ of strictly convergent power series in several variables with its formal completion $L[[x_1, \dots, x_n]]$. see [2] or [12];

$$\text{Henselisation}(L[x_1, \dots, x_n]_{(x_1, \dots, x_n)}) = L[[x_1, \dots, x_n]] \cap L(x_1, \dots, x_n)^{alg}$$

This implies that

$$O_{\bar{0}, A^n}^\wedge \cong L[[x_1, \dots, x_n]] \cap L(x_1, \dots, x_n)^{alg}$$

$$O_{a,F}^\wedge \cong L[[w_1, \dots, w_n]] \cap L(w_1, \dots, w_n)^{alg} \quad (*)$$

We now use analytic results for the formal power series ring $L[[w_1, \dots, w_n]]$. By Weierstrass preparation, we obtain the equations

$$x_1 = u_1(w_1^{m_1} + q_{11}(w_2, \dots, w_n)w_1^{m_1-1} + \dots q_{m_1 1}(w_2 \dots w_n))$$

$$x_2 = u_2(w_1^{m_2} + q_{12}(w_2, \dots, w_n)w_1^{m_2-1} + \dots q_{m_2 2}(w_2 \dots w_n))$$

...

$$x_n = u_n(w_1^{m_n} + q_{1n}(w_2 \dots w_n)w_1^{m_n-1} + \dots q_{m_n n}(w_2 \dots w_n)) \quad (**)$$

where the u_i are units in $L[[w_1, \dots, w_n]]$ and the q_{ij} are polynomials without constant term. In order to apply Weierstrass preparation, we require that the power series expansions for the x_i should be regular with respect to the variable w_1 . Clearly this can be achieved in the following manner;

Let $M = (m_{kl})_{1 \leq k, l \leq n}$ be an invertible matrix of elements in L . Then if $\bar{w}' = M(\bar{w})$, as M is invertible, \bar{w}' is also a set of uniformisers for $L[[w_1, \dots, w_n]]$. The condition of irregularity C_{ij} for x_i in terms of the variable w'_j is a (possibly infinite) conjunction of closed relations on the m_{kl} . Hence there exists a Zariski open set $U \subset GL_n(L)$ such that C_{ij} fails to hold for $1 \leq i, j \leq n$, that is, after a linear change of variables, we can assume that the x_i each have regular expansions in terms of w_j .

Now w_1, \dots, w_n are algebraically independent in $L(F)$ which has transcendence degree n over L . As each $x_i \in L(F)$, we must have that each $x_i \in L(w_1, \dots, w_n)^{alg}$. Therefore, the u_i in the equations $(**)$ can be taken in $L(w_1, \dots, w_n)^{alg}$ and, using $(*)$, the equations hold in $O_{a,F}^\wedge$.

(3) Hence, we can find an etale morphism $\pi : (U, b) \rightarrow (F, a)$ such that the equations $(**)$ hold in the local ring $O_{U,b}$. Again, we may assume that U is irreducible and smooth.

(4)-(7) This part of the argument goes through essentially unchanged, with the slight modification that the projective closure F' of U may fail to be smooth and the closure of $graph(\pi)$ in $F' \times F$ may fail to define a function, only a generically finite correspondence between F' and F . However, this still allows us to work in the context of Theorem

3.3, see also the Remarks 3.4, when we consider the projection of the correspondence restricted to U .

(8) Now we have the sequence of ring inclusions given by

$$L[x_1, \dots, x_n] \rightarrow L[\bar{x}, \bar{w}, \bar{u}] / \langle x_1 - u_1 p_1(\bar{w}), \dots, x_n - u_n p_n(\bar{w}), s_1(u_1), \dots, s_n(u_n) \rangle \rightarrow R$$

where R is the coordinate ring of U in some affine neighborhood of b , p_i are the polynomials given in (***) and s_i are the minimum polynomials of u_i over $L(w_1, \dots, w_n)$. A simple calculation shows that the second variety is smooth at $\bar{0}$ and the second inclusion corresponds to an etale extension of algebras. It is therefore sufficient to check that the algebraic and Zariski multiplicities of the left hand inclusion coincide at $\bar{0}$ (***). An easy calculation gives that the algebraic multiplicity of the left hand inclusion is $\text{length}(L[\bar{w}, \bar{u}]_{\bar{0}} / \langle u_1 p_1(\bar{w}), \dots, u_n p_n(\bar{w}), s_1(u_1), \dots, s_n(u_n) \rangle)$ which, by the localisation at $\bar{0}$, is just $\text{length}(L[\bar{w}]_{\bar{0}} / \langle p_1(\bar{w}), \dots, p_n(\bar{w}) \rangle)$. This is precisely the intersection multiplicity of the hypersurfaces p_1, \dots, p_n at $\bar{0}$. Again, for ease of exposition, we compute the case of 2 irreducible intersecting polynomials $p_1(x, y) = 0$ and $p_2(x, y) = 0$ with $p_1(0, 0) = p_2(0, 0) = 0$. We claim the following theorem;

Theorem 6.10. *The intersection multiplicity of p_1, p_2 at $(0, 0)$ corresponds to the Zariski multiplicity of the cover $\text{Spec}(L[xyuv]) / \langle p_1 - u, p_2 - v \rangle \rightarrow \text{Spec}(L[uv])$, when $\text{Char}(L) = 0$.*

The theorem includes the proof of (***) when $n = 2$. We shall indicate how the higher dimensional case follows later. In order to prove the theorem, we need a series of lemmas.

Lemma 6.11. *Let $F(x, \bar{y})$ be an irreducible Weierstrass polynomial in x with $F(0, \bar{0}) = 0$ then algebraic multiplicity and Zariski multiplicity coincide for the cover $\text{Spec}(L[x\bar{y}] / \langle F \rangle \rightarrow \text{Spec}(L[\bar{y}])$.*

Proof. We have that $F(x, \bar{y}) = x^n + q_1(\bar{y})x^{n-1} + \dots + q_n(\bar{y})$ where $q_i(\bar{0}) = 0$. The algebraic multiplicity is given by $\text{length}(L[x] / F(x, \bar{0})) = \text{ord}(F(x, \bar{0})) = n$ in the ring $L[x]$ with the canonical valuation. We first claim that the Zariski multiplicity is the number of solutions to $x^n + q_1(\bar{e})x^{n-1} + \dots + q_n(\bar{e}) = 0$ (*), where \bar{e} is generic in $\mathcal{V}_{\bar{0}}$. For suppose that (a, \bar{e}) is such a solution, then $F(a, \bar{e}) = 0$ and by specialisation $F(\pi(a), \bar{0}) = 0$. As F is a Weierstrass polynomial in x , $\pi(a) = 0$, hence $a \in \mathcal{V}_0$, giving the claim. As $\text{char}(L) = 0$, $\text{Disc}(F(x, \bar{y})) = \text{Res}_{\bar{y}}(F, \frac{\partial F}{\partial x})$

is a regular polynomial in \bar{y} defined over L . By genericity of $\bar{\epsilon}$, we have that $Disc(F(x, \bar{y}))|_{\bar{\epsilon}} \neq 0$, hence $(*)$ has no repeated roots. This gives the lemma. \square

Lemma 6.12. *Let $F(x, \bar{y})$ be an irreducible polynomial with $F(x, \bar{0}) \neq 0$ and $F(0, \bar{0}) = 0$. Then the Zariski multiplicity of the cover $Spec(L[x, \bar{y}]/<F>) \rightarrow Spec(L[\bar{y}])$ equals $ord(F(x, \bar{0}))$ in $L[x]$.*

Proof. By the Weierstrass Preparation Theorem, we can write $F(x, \bar{y}) = U(x, \bar{y})G(x, \bar{y})$ with $U(x, \bar{y}), G(x, \bar{y}) \in L[[x, \bar{y}]]$, $G(x, \bar{y})$ a Weierstrass polynomial in x and $deg(G) = ord(F(x, \bar{0}))$. As above, we may take the new coefficients to lie inside the Henselized ring $L[x, \bar{y}]_0^\wedge$, hence inside some finite etale extension $L[x, \bar{y}]^{ext}$ of $L[x, \bar{y}]$ (possibly after localising $L[x, \bar{y}]$). Now we have the sequence of morphisms;

$$Sp(L[x, \bar{y}]^{ext}/UG) \rightarrow Spec(L[x, \bar{y}]/F) \rightarrow Spec(L[\bar{y}])$$

The left hand morphism is etale at $\bar{0}$, hence as we have seen, to compute the Zariski multiplicity of the right hand morphism, we need to compute the Zariski multiplicity of the cover

$$Spec(L[x, \bar{y}]^{ext}/UG) \rightarrow Spec(L[\bar{y}])$$

Choose $\bar{\epsilon} \in \mathcal{V}_{\bar{0}}$, the fibre of the cover is given formally analytically by $L[[x, \bar{y}]]/<UG> \otimes_{L[\bar{y}], \bar{y} \mapsto \bar{\epsilon}} L$, hence by solutions to $U(x, \bar{\epsilon})G(x, \bar{\epsilon})$. By definition of Zariski multiplicity, we consider only solutions $(x\bar{\epsilon})$ in $\mathcal{V}_{(0, \bar{0})}^{lift}$, (here $(0, \bar{0})^{lift}$ is the lift of $(0, \bar{0})$ in the etale neighborhood, for ease of notation we will just use $(0, \bar{0})$ from now on.) As $U(x, \bar{y})$ is a unit in the local ring $L[x, \bar{y}]_{0, \bar{0}}^{ext}$, we must have $U(x, \bar{\epsilon}) \neq 0$ for such solutions. Hence, the solutions are given by $G(x, \bar{\epsilon}) = 0$. Now, we use the previous lemma to give that the Zariski multiplicity is exactly $deg(G)$ as required. \square

Lemma 6.13. *Let $p_1(x, y), p_2(x, y)$ be Weierstrass polynomials in x with $p_1(0, 0) = p_2(0, 0) = 0$. Then the Zariski multiplicity of the cover $Spec(L[x, y, u, v]/<p_1 - u, p_2 - v>) \rightarrow Spec(L[u, v])$ $(*)$ at $(\bar{0})$ equals the intersection multiplicity at $(0, 0)$, $I(p_1, p_2, (0, 0))$.*

Proof. Let $F(y, u, v) = \text{Res}(p_1 - u, p_2 - v)$. Then $F(0, 0, 0) = \text{Res}(p_1, p_2)(0) = 0$, as p_1, p_2 have a common root at $(0, 0)$. By a result due to Abhyankar, see for example [1], $\text{ord}_y(F(y, \bar{0})) = \Sigma I(p_1, p_2, (x0))$ at common solutions $(x, 0)$ to p_1 and p_2 over 0 . As p_1 and p_2 are Weierstrass polynomials in x , this is just $I(p_1, p_2, (00))$. By the previous lemma, it is therefore sufficient to prove that the Zariski multiplicity of the cover $(*)$ at $(0, 0, 0, 0)$ equals the Zariski multiplicity of the cover $\text{Spec}(K[y, u, v]/\langle F \rangle \rightarrow \text{Spec}(K[u, v])$ $(**)$ at $(0, 0, 0)$. Suppose the Zariski multiplicity of $(**)$ equals n . Then there exist $y_1, \dots, y_n \in \mathcal{V}_0$ distinct and $\bar{\epsilon} \in \mathcal{V}_{00}$ such that $F(y_i, \bar{\epsilon})$ holds. Consider $Q(u, v) = \text{res}(F(y, u, v), \frac{\partial F}{\partial y}(y, u, v))$. By genericity, we have that $Q(\bar{\epsilon}) \neq 0$. Hence, $F(y_i, \bar{\epsilon})$ is a non-repeated root. Using Abhyankar's result, we can find a unique x_i with $(x_i y_i)$ a common solution to $p_1 - \epsilon_1$ and $p_2 - \epsilon_2$. We claim that each $(x_i y_i) \in \mathcal{V}_{00}$. As $p_1(x_i y_i) - \epsilon_1 = 0$, by specialisation $p_1(\pi(x_i), 0) = 0$. Now, using the fact that p_1 is a Weierstrass polynomial in x , gives that $\pi(x_i) = 0$ as well. This shows that the Zariski multiplicity of the cover $(*)$ is at least n . A virtually identical argument shows that the Zariski multiplicity of the cover $(*)$ is at most n as well. This gives the result. □

Lemma 6.14. *Let $p_1(x, y), p_2(x, y)$ be polynomials with $p_1(0, 0) = p_2(0, 0) = 0$. Then the Zariski multiplicity of the cover $\text{Spec}(L[xyuv]/\langle p_1 - u, p_2 - v \rangle \rightarrow \text{Spec}(L[uv])$ equals $I(p_1, p_2, (00))$.*

Proof. Again, using the Weierstrass Preparation Theorem, write $p_1(x, y) = u_1(x, y)f_1(x, y)$ and $p_2(x, y) = u_2(x, y)f_2(x, y)$, with f_1, f_2 Weierstrass polynomials in x . As before, we may assume the new coefficients lie in a finite ring extension $L[x, y]^{\text{ext}}$ such that the map

$$\text{Spec}(L[x, y]^{\text{ext}}[u, v]/\langle u_1 f_1(x, y) - u, u_2 f_2(x, y) - v \rangle \rightarrow \text{Spec}(L[xyuv]/\langle f_1 - u, f_2 - v \rangle)$$

is etale near $\bar{0}$

Again, it is sufficient to prove that the Zariski multiplicity of the cover $\text{Spec}(L[x, y]^{\text{ext}}[u, v]/\langle u_1 f_1(x, y) - u, u_2 f_2(x, y) - v \rangle \rightarrow \text{Spec}(L[u, v])$ at $(0, 0, 0, 0)$ equals $I(u_1 f_1, u_2 f_2, 00) = I(f_1, f_2, 00)$. For this we need the following “unit removal” lemma.

Lemma 6.15. (*Unit Removal*)

Let $u_1(x, y), u_2(x, y), f_1(x, y), f_2(x, y)$ be polynomials in $L[x, y]$ with u_1, u_2 units in the local ring $L[x, y]_{0,0}$. Then the Zariski multiplicity of the cover $\text{Spec}(L[x, y, u, v] / \langle u_1 f_1(x, y) - u, u_2 f_2(x, y) - v \rangle \rightarrow \text{Spec}(L[u, v])$ (*) is equal to the Zariski multiplicity of the cover $\text{Spec}(L[x, y, u, v] / \langle f_1(x, y) - u, f_2(x, y) - v \rangle \rightarrow \text{Spec}(L[u, v])$ (**).

In order to prove the lemma, we first need to introduce a new version of Zariski multiplicity. Suppose that $F \subset D \times V^n$ is a finite cover of a smooth 2-dimensional base D .

Definition 6.16. Given $(a, \lambda_1, \lambda_2) \in F$, we define;

Left.Mult $_{a, \lambda_1, \lambda_2}(F/D) = \text{Card}(\mathcal{V}_a \cap F(x, \lambda'_1, \lambda_2))$ for $\lambda'_1 \in \mathcal{V}_{\lambda_1}$ generic over L .

Right.Mult $_{a, \lambda_1, \lambda_2}(F/D) = \text{Card}(\mathcal{V}_a \cap F(x, \lambda_1, \lambda'_2))$ for $\lambda'_2 \in \mathcal{V}_{\lambda_2}$ generic over L .

By factoring the specialisations involved, it is easily shown that both left multiplicity, right multiplicity are well defined and moreover the following holds;

$$\text{Mult}_{(a, \lambda_1, \lambda_2)}(F/D) = \sum_{a' \in (\mathcal{V}_a \cap F(x, \lambda'_1, \lambda_2))} \text{Right.Mult}_{(a' \lambda'_1 \lambda_2)}(F/D)$$

$$\text{Mult}_{(a, \lambda_1, \lambda_2)}(F/D) = \sum_{a' \in (\mathcal{V}_a \cap F(x, \lambda_1, \lambda'_2))} \text{Left.Mult}_{(a' \lambda_1 \lambda'_2)}(F/D)$$

That is we may compute the Zariski multiplicity by varying the family in 2 stages. Now, in the case of the lemma, after varying one parameter, an easy algebraic calculation shows the resulting curves intersect transversally at simple points $(x_i y_i)$. In this case we can apply the inverse function theorem to one curve C_1 given by $u_1 f_1 = 0$ and obtain formally analytic presentations around each $(x_i y_i)$ in the variable t_i . As we have already seen in the previous use of analytic methods, this does not effect the calculation of Zariski multiplicity. If $(t_i, h(t_i))$ with $h(t_i) \in L[[t_i]]$ is a local analytic presentation of C_1 at $(x_i y_i)$, then, by transversality, we have $\text{ord}_{t_i}(u_2 f_2(t_i, h(t_i))) = 1$ and we have to check that this agrees with the Zariski Right multiplicity. This calculation has already been done in Theorem 6.5. Hence, we can calculate the Zariski multiplicity of (*) and (**) as the Zariski Left multiplicity. Now, we claim that the Zariski Left multiplicity of the covers (*) and (**) is the same. This is a straightforward calculation, suppose that

the Zariski Left Multiplicity of $(*)$ is n . Then there exists ϵ generic and $(x_1y_1), \dots, (x_ny_n) \in \mathcal{V}_0$ such that $u_1q_1(x_iy_i) = \epsilon$ and $u_2q_2(x_iy_i) = 0$. Now using the fact that the u_i are units, we find ϵ' generic in \mathcal{V}_0 such that $q_1(x_iy_i) = \epsilon'$ and $q_2(x_iy_i) = 0$. This shows exactly that the Zariski Left Multiplicity of $(**)$ at $(0, 0, 0, 0)$ is at least n . Reversing the argument shows the Zariski Left Multiplicity is exactly n as required

Now the proof of Lemma 6.13 follows from the proof of Lemma 6.12. \square

Higher dimensional case; The same method as for curves, inductive argument using Abhyankar's Lemma on resultants and Weierstrass Preparation for the ring $L[[x_1, \dots, x_nx_{n+1}]]$.

7. FURTHER DIRECTIONS OF STUDY

Remarks 7.1. *Let hypotheses be as in Theorem 3.3, with the additional assumption that F is an etale cover of $D = A^n$. Then one can improve the lifting condition to points in $L[[t]]$. Use the local uniformisers to present the cover over A^n in the form $f_1(\bar{x}, \bar{y}) = 0, f_2(\bar{x}, \bar{y}) = 0, \dots, f_n(\bar{x}, \bar{y}) = 0$, with \bar{x}, \bar{y} tuples in A^n . Then letting \bar{x} be a point in $L[[t]]$ gives n equations inside A^n with coefficients in $L[[t]]$. By Hensel's lemma, we can find a solution to these equations in $L[[t]]$, as reducing the equations modulo (t) , by the fact that the morphism is etale at \bar{a} , $\bar{f}_1, \dots, \bar{f}_n$ have a common solution \bar{a} in L with $(\frac{\partial \bar{f}_i}{\partial y_j})_{ij}(\bar{a}) \neq 0$*

In deformation theory arguments, we work with schemes defined over the projective limit of rings $L[t]/(t^n)$. This suggests developing part of the theory of Zariski structures in the analytic context of complete valued fields, possibly using the Pas language with sorts for the reductions modulo t^n . We save this point of view for another occasion.

Remarks 7.2. *As mentioned before, one can define the etale topology and obtain the Cech cohomology groups with finite coefficients for any 1-dimensional Zariski structure. The following is a classical result, most famously used in Deligne's proof of the Weil conjectures;*

(Lefschetz fixed-point formula)

Let X be a complete non-singular variety over an algebraically closed field K , and let $\phi : X \rightarrow X$ be a regular map. Then

$$(\Gamma_\phi \cdot \Delta) = \Sigma(-1)^r \text{Tr}(\phi|H^r(X, \mathcal{Q}_l)) (*)$$

where Γ_ϕ is the graph of ϕ , Δ is the diagonal in $X \times X$ and $(\Gamma_\phi \cdot \Delta)$ is the number of fixed points of ϕ counted with multiplicity.

Now both sides of the above formula make sense in the more generalised setting of X , a closed presmooth subset of C^n , where C is a 1-dimensional Zariski structure. We use the notion of Zariski multiplicity to replace algebraic multiplicity. The natural question is the following;

For what class of Zariski structures does equality hold in $(*)$?

In the algebraic context, the Lefschetz formula is a formal consequence of a cohomology theory with good properties;

1. Kunneth Formula.
2. Finite dimensionality of the groups $H^i(X, F_l^n)$ for prime l .
3. Poincare duality.
4. Existence of a cycle map $\text{cl}_X^* : CH^*(X) \rightarrow H^*(X)$
5. Smooth and Proper Base Change Theorems.

(Here $CH^*(X)$ is the graded Chow ring of cycles on X and $H^*(X)$ is the graded cohomology ring on X .)

Clearly, an answer to the above can be reduced to further questions concerning the class of Zariski structures for which the properties 1-5 hold. The interested reader should look at [8] or [3]

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